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PROVING INVERSE-POSITIVITY OF LINEAR OPERATORS BY REDUCTION

Johann Schroeder

Boeing Scientific Research Laboratories
Seattle, Washington

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PROVING INVERSE-POSITIVITY OF LINEAR OPERATORS BY REDUCTION

by

Johann Schröder*

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*Visiting from the University of Cologne, Germany.

Abstract

A method is presented to prove that a linear operator M is inverse-positive, i.e. $Mu \geqq o$ implies $u \geqq o$. The method consists in reducing the problem for the given operator M to a simpler problem. Sometimes, iterated reductions are appropriate, for example, if M is a differential operator of higher order.

1. Introduction.

This paper provides tools for proving that a linear operator M is *inverse-positive*, that means: $Mu \geq o$ *implies* $u \geq o$. This property is frequently used in Applied Mathematics. For example, it is closely related to the boundary maximum principle for second order differential operators.

The abstract formulation of the above property, and a corresponding property for nonlinear operators, is due to Collatz (see [1] where such operators M are called "Operatoren monotoner Art", i.e. operators of monotone type). Collatz showed the importance of this property in Numerical Mathematics and, in particular, for obtaining error bounds.

An abstract theory on such operators provides sufficient and necessary conditions (see [2] and the references given there). In this paper, we develop methods for proving that these sufficient conditions are satisfied in more complicated cases where this is not immediately seen, for example, in case of higher order differential operators.

The provided methods reduce the problem of proving that M is inverse-positive to a corresponding problem for an operator $\hat{M} = LM$ (or $\hat{M} \geq LM$). In some cases, a suitable operator L can immediately be constructed (Section 3); in others, several operators are used which theoretically can be combined in a single operator, but practically will not (Section 4). This last method is applied to operators $M = I-B$ with positive B, in Section 5. There are cases, when it is appropriate to actually carry out several reductions, one after the other (Section 6). For example, the order of differential operators can be reduced in this way, step by step.

2. Notations and Basic Assumptions.

Let R and S denote real linear spaces and $M: R \to S$ a linear operator. Suppose that in both these spaces there are defined order relations, both denoted by $\leq$.

All occurring order relations shall be reflexive, transitive and compatible with the linear structure. The space $(R,\leq)$ will be called *partially ordered* if the relation $\leq$ is also antisymmetric. By K_R we denote the cone of all $u \geq o$ in R. For u in the algebraic interior (core) of K_R, we will write $u \succ o$. (In other words, $u \succ o$ if and only if, for each $v \in R$, there exists some number n with $nu + v \geq o$.) Corresponding notations are used for S. For all occurring order relations, $> o$ shall be equivalent to $\geq o$ and $\neq o$, so that, for example, $u \succeq o$ means that either $u \succ o$, or $u = o$.

We assume that $(R,\leq)$ is *Archimedian* (that means K_R is *linearly closed*, or: for all $u,v \in R$, $nu + v \geq o$ $(n=1,2,\ldots)$ implies $u \geq o$). Suppose, moreover, that there is defined a second order relation $\boldsymbol{\leq}$ in S which dominates the first in the following sense: For $U,V \in S$,

$$U \boldsymbol{\geq} o \Longrightarrow U \geq o; \quad U \boldsymbol{>} o, \quad V \geq o \Longrightarrow U + V \boldsymbol{>} o.$$

We are then interested in the following properties of M:

IP $\quad Mu \geq o$ *implies* $u \geq o$, *for* $u \in R$.

IP' $\quad Mu \boldsymbol{>} o$ *implies* $u \succ o$, *for* $u \in R$.

The operator M is called *inverse-positive* if **IP** holds. *If* R *is partially ordered and* M *has both properties,* **IP** *and* **IP'**, *then*

$$Mu \boldsymbol{\geq} o \text{ implies } u \succeq o, \text{ for } u \in R.$$

For, in this case, $Mu = o$ can only occur if $u = o$.

In case, $\succ$ is equivalent to $>$, the last property will be called *strong inverse-positivity*:

SIP $\quad Mu \geqq o$ *implies* $u \succeq o$, *for* $u \in R$.

Theorem 1 in [2] contains sufficient conditions **I, II** for **IP** which can also be used for proving **IP'**. This paper is mainly concerned with providing methods for proving

I $\quad Mv \nleqq o$ *for each* $v \in R$ *with* $v \geq o$, $v \nsucc o$.

For describing these methods, we will need also other linear spaces with two order relations $\leqq$ and $\preceq$ such that the second dominates the first. The letter X (also with subscript, etc.) will always denote such space. For these spaces, we will use corresponding notations as for R (K_X, $x \succ o$, etc.).

Furthermore, the letter L will always denote a linear operator mapping such space X (or S) into another such space $\hat{X}$ and satisfying the following two conditions. For each $x \in X$ ($x \in S$),

(2.1) $\qquad x \geqq o$ *implies* $Lx \geqq o$,

(2.2) $\qquad x \succ o$ *implies* $Lx \succ o$.

(For some of the following results only one or the other of these properties is needed. Notice also that (2.1) follows from (2.2) if $(\hat{X}, \leqq)$ is Archimedian and if there exists an element $x \succ o$ in X.)

3. Simple Reduction

The following simple theorem is the basic tool for what is derived in this paper.

3.1 Theorem. *Let there exist a linear operator* L *mapping* S *into a linear space* X *such that* (2.1), (2.2) *hold and the following conditions are satisfied:*

I' $LMv \not> o$ *for each* $v \in R$ *with* $v \geq o$, $v \not\succ o$.

II' *There exists an element* $z \in R$ *with*

$$(3.2) \qquad z \geq o, \quad LMz > o.$$

Then, M *has the Properties* IP *and* IP'.

Proof. Since $Mu \geq o$ implies $LMu \geq o$ and $Mu > o$ implies $LMu > o$, it suffices to prove the Properties IP and IP' for LM, instead of M. The above assumptions, however, are equivalent to the assumptions I, II in Theorem 1 [2], with M replaced by LM. According to that theorem, LM therefore has Property IP. This property together with I' yields that, $LMu > o$ implies $u \succ o$.

Remark. *Instead of* II' *one can require:*

II *There exists an element* $z \in R$ *with* $z \geq o$, $Mz > o$.

This follows from (2.2).

The above theorem leads to the following

Method of Reduction: *Find an operator* $L: S \to X$ *satisfying* (2.1), (2.2) *such that* $LM = \hat{M} - N$ *where* $\hat{M}$ *has Property* **I** *and* N *is positive* ($u \geq o \Longrightarrow Nu \geq o$).

If then **II'** *or* **II** *holds,* M *has Properties* **IP** *and* **IP'**.

The last statement follows from the fact that **I'** is satisfied if $\hat{M}$ has Property **I**.

With this method, the problem involving M is reduced to a problem with the operator $\hat{M}$. This can have the following *two* advantages. First, $\hat{M}$ may be simpler, and second, Property **I** for $\hat{M}$ may be weaker than Property **I** for M. Notice, that Property **I** depends on the order relation $<$ defined in the range of the operator so that different such order relations belong to M and $\hat{M}$, respectively.

Example. Let $R = X = \mathbb{R}^n$, $S = \mathbb{R}^m$ and identify operators with matrices. Suppose that $\leq$ denotes the natural (componentwise) order relation, in each of these spaces. For $U \in S$, define $U \mathbf{>} o$ by, $U \succ o$ in case (i), $U > o$ in case (ii). For $x \in X$, $x \mathbf{>} o$ and $x \succ o$ shall be equivalent.

Then, an $n \times m$-matrix L has the properties (2.1), (2.2) if and only if in case (i): each row of L contains an element > 0, in case (ii): all elements of L are > 0.

If the matrix $\hat{M}$ is diagonal, it has Property **I** (as an operator: $R \to X$). Thus, if we can find a matrix L of the prescribed type such that all off-diagonal elements of LM are ≤ 0, the matrix

M has Properties $\mathbb{P}$ and $\mathbb{P}'$ (as an operator: $R \to S$), provided there exists a suitable vector z.

For example, we have

$$LM = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 5 \end{bmatrix} \quad \text{for } M = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -3 & 3 & -1 & 0 \\ 1 & -3 & 3 & -1 \\ 0 & 1 & -3 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Since L has all elements > 0 we consider case (ii). Then, z with $z^T = (1,2,3,4)$ satisfies (3.2). Therefore, M has Property $S\mathbb{P}$ which implies that M possesses an inverse with all elements > 0.

4. Reducing with Several Operators

When a suitable operator L as needed in Theorem 3.1 cannot immediately be found the following result can be helpful.

4.1 Theorem. *Let the operator* M *have the following two properties.*

$\mathbb{I}''$ *For each* $v \in R$ *with*

$$v \geq o, \quad v \not> o \tag{4.2}$$

there exists a linear operator $L = L(v)$ *mapping* S *into a linear space* $X = X(v)$ *such that* (2.1) *and* (2.2) *hold, and*

$$LMv \not> o. \tag{4.3}$$

$\mathbb{II}$ *There exists an element* $z \in R$ *such that* $z \geq o$, $Mz > o$.

Then, M *has the Properties* $\mathbb{P}$ *and* $\mathbb{P}'$.

Remark. *The theorem remains true if* $\mathbb{II}$ *is replaced by,*

$\mathbb{II}''$ *There exists an element* $z \in R$ *with*

$z \geq o$, $L(v)Mz > o$ *for all operators* $L(v)$ *occurring in* $\mathbb{I}''$.

<u>Proof of Theorem and Remark</u>. Let $\mathcal{V}$ be the set of all elements $v \in R$ satisfying $v \geq o$, $v \not> o$ (i.e. $\mathcal{V} = \partial K$). We introduce the product space $\hat{X} = \prod_{v \in \mathcal{V}} X(v)$ with elements $\hat{x} = \{x(v)\}$ and define the order relations $\hat{x} \geq o$ and $\hat{x} > o$ componentwise. Moreover, let the operator $\hat{L}: S \to \hat{X}$ be defined by $\hat{L}U = \{L(v)U\}$. This operator $\hat{L}$, instead of L, has the properties (2.1), (2.2). Moreover, the assumptions I'' and II'' imply that the conditions I', II' of Theorem 3.1 are satisfied with L replaced by $\hat{L}$. Since II'' is a consequence of II, the statements to be proven follow from Theorem 3.1.

If $X(v)$ is chosen to be $\mathbb{R}$ for all v and there exists a $U > o$ in S, I'' obtains the following form when we write f for L.

I''' *For each* $v \in R$ *satisfying* $v \geq o$, $v \not> o$, *there exists a linear functional* f *on* S *such that*

(4.4) $\quad fU > 0$ *for all* $U > o$, *and* $fMv \leq 0$.

This property has been applied earlier [4], and generalized to nonlinear problems [3]. In many cases, Property I''' is equivalent to I which can be shown by using separation theorems.

Suppose, that the set Y *of all* $U > o$ *has the following property: Whenever a point* $V \in S$ *is not in* Y, *then there exists a linear functional* f *on* S *such that* $fU > 0$ *for all* $U \in S$, *and* $fV \leq 0$.

Under this condition, I *and* I''' *are equivalent.*

In order to show that I implies I''', let $v \geq o$ and $v \not> o$. Then, $Mv \not> o$ so that (4.4) follows from the above assumptions using $V = Mv$.

4.5 Example. Let L denote the differential operator, defined by $Lu = -u^{VI} + qu$ for $u \in C_6[0,1]$ with given $q \in C[0,1]$, and consider the boundary conditions,

$$\begin{aligned} &u(0) = 0 \quad , && u(1) = 0 \quad , \\ &u^{IV}(0) - \alpha u'(0) = 0 \quad , && u^{IV} + au'(1) = 0 \quad , \\ &u^{V}(0) + \beta u''(0) + \gamma u'(0), && -u^{V}(1) + bu''(1) - cu'(1) = 0. \end{aligned}$$

We assume that

$$q(s) \leq 0 \quad (0 \leq s \leq 1), \quad \alpha \geq 0, \quad a \geq 0, \quad \beta \geq 0, \quad b \geq 0.$$

4.6 *Let there exist a function* $z \in C_6[0,1]$ *satisfying the given boundary conditions such that* $z(s) \geq 0$, $(Lz)(s) \geq 0$ *and* $(Lz)(s) \not\equiv 0$ $(0 \leq s \leq 1)$. *Then, the following is true.*

If $u \in C_6[0,1]$ *satisfies the given boundary conditions and* $L[u](s) \geq 0$ $(0 \leq s \leq 1)$, *then* $u(s) \geq 0$ $(0 \leq s \leq 1)$, *and even* $u \succeq o$ *with the order relation* $\succeq$ *defined below (in (4.7)).*

We will prove this statement by applying Theorem 4.1. Define $S = C[0,1]$ and let R be the set of all $u \in C_6[0,1]$ which satisfy the given boundary conditions. In both spaces, R and S, $\leq$ shall denote the natural (pointwise) order relation. Then, for $u \in R$,

(4.7) $u \succ o$ if and only if $u(s) > 0$ $(0 < s < 1)$, $u'(0) > 0$, $u'(1) < 0$.

For $U \in S$, let $U \geqq o$ be equivalent to $U \geq o$. Moreover, define M to be the restriction of L from C_6 to R. We have then to prove Property I" for this operator.

Let $v \geq 0$, $v \not> 0$ for some $v \in R$. Then either $v'(0) = 0$, or $v'(1) = 0$, or $v(\tau) = 0$ for some $\tau \in (0,1)$, or several of these relations hold. For each such v, we have to find a suitable operator $L = L(v)$ mapping S into a space $X = X(v)$.

In case $v'(0) = 0$ we choose $X = \mathbb{R}$, and the linear functional $LU = \int_0^1 f(s)U(s)ds$ with $f(s) = 1-s$. Obviously, (2.1) and (2.2) are true (for $x \in \mathbb{R}$, $x > 0$ shall be equivalent to $x > 0$). By partially integrating twice, using the boundary conditions and observing $v''(0) \geq 0$, $v'(1) \leq 0$, we obtain $LMv \leq 0$. Thus, (4.3) has been proved in this case.

In case $v'(1) = 0$ we proceed similarly using the functional $LU = \int_0^1 g(x)U(s)ds$ with $g(s) = s$.

Consider now the third case,

(4.8) $v(\tau) = 0$ for some $\tau \in (0,1)$, $v'(0) > 0$, $v'(1) < 0$.

We calculate AMv with the operator A defined by

$$(AU)(t) = \int_0^1 K(s,t)U(s)ds, \quad K(s,t) = \begin{cases} f(s)g(t) & \text{for } 0 \leq t \leq s \leq 1, \\ f(t)g(s) & \text{for } 0 \leq s \leq t \leq 1, \end{cases}$$

and again, $f(t) = 1-t$, $g(t) = t$. Using similar means as above, we obtain $(AMv)(t) \leq v^{IV}(t)$ $(0 \leq t \leq 1)$. In order to reduce the order of the occurring derivatives further, we apply a second integral operator

$$(Bx)(r) = \int_0^1 G(t,r)x(t)dt, \quad G(t,r) = \begin{cases} \Phi(t)\Psi(r) & \text{for } 0 \leq r \leq t \leq 1 \\ \Phi(r)\Psi(t) & \text{for } 0 \leq t \leq r \leq 1 \end{cases}$$

and $\Phi(t) = (1-t)^2$, $\Psi(t) = t^2$. By again partially integrating twice and observing $v'(\tau) = 0$, $v''(\tau) \geq 0$, we obtain

$(BAMv)(\tau) \leqq (Bv^{IV})(\tau) =$

$$-2\tau(1-\tau)v''(\tau) + 2(1-2\tau)v'(\tau) - 2(1-\tau)^2v'(0) + 2\tau^2v'(1) < 0.$$

For v satisfying (4.8), we define $X = C_0[0,1]$ with $x \geq o$ equivalent to the natural order relation, and $L = BA$. Then, (2.1) and (2.2) hold, and (4.3) is also true.

This proves the statement.

5. Application to Operators $M = I-B$ with Positive B.

Let now the sets R and S be equal and $\leqq$ denote the same order relation, in R and S. We have, moreover, the order relations $\leqslant$ in $S = R$, and $\preceq$ in $R = S$. We write M as $M = I-B$ with the unit operator I and ask for conditions on the operator $B: R \to R$ such that M has Properties $\mathbb{P}$ and $\mathbb{P}'$.

5.1 Theorem. *Suppose that the following three conditions are satisfied (for all* $u \in R$*):*

(i) $u \geqq o$ *implies* $Bu \geqq o$ *(B is positive).*

(ii) $\left.\begin{array}{l} u \geqq o \\ u > B^n u \quad (n=1,2,\dots) \end{array}\right\}$ *implies* $u \succ o$.

(iii) *There exists an element* $z \in R$ *such that* $z \geqq o$, $(I-B)z > o$.

Then, the operator $M = I-B$ *has the Properties* $\mathbb{P}$ *and* $\mathbb{P}'$.

Proof. Condition (ii) implies that for each $v \in R$ with $v \geqq o$, $v \not\succ o$, there exists a natural number n such that $(I-B^n)v \not> o$.

Define then $L = L(v) = I + B + \cdots + B^{n-1}$. This operator maps $S = R$ into $X = R$ and satisfies $LMv = (I-B^n)v \succ o$ as well as (2.1) and (2.2). Thus, $M = I-B$ has Property I" of Theorem 4.1, and the statements to be proven here follow from that theorem.

For positive B, *the assumption* (ii) *is satisfied if* (a) *or* (b) *holds:*

(a) $u > o$ *is equivalent to* $u \succ o$.

(b) $u > o$ *implies* $B^n u \succ o$ *for some natural number* n.

Condition (b) in general requires that some power of B has a stronger positivity property. In view of Property IP' (and SIP) we like the second order relation $\leqq$ in S to be as weak as possible. The weaker this relation is, the stronger is condition (b).

Example. Let $R = \mathbb{R}^m$ and $\leq$ denote the natural (componentwise) order relation. If $u > o$ is equivalent to $u > o$, property (b) means that some power B^n of the matrix B (the matrix associated with the operator B) has all its elements > 0. An example is

$$B = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{with} \quad B^2 = \frac{1}{16}\begin{bmatrix} 5 & 4 & 1 \\ 4 & 6 & 4 \\ 1 & 4 & 5 \end{bmatrix}.$$

Other definitions of $u > o$ also make sense. Let $R = \mathbb{R}^p \times \mathbb{R}^q$ with natural numbers $p, q, p+q=m$, and partition $u \in R$ correspondingly: $u^T = (u_1^T, u_2^T)$. Define then $u > o$ by $u_1 > o$, $u_2 > o$. In case $p = 2$, $q = 2$, the matrix

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \quad \text{with} \quad B_{11} = B_{22} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

satisfies (b) with $n = 1$.

Inverse-positivity of $M = I-B$ with positive B can also be proven by different means. For example, if the series $I + B + B^2 + \cdots$ converges to $(I-B)^{-1}$, in some sense to be defined, the positivity of the limit operator in general follows from the positivity of B. Under suitable assumptions (using topological terms) this convergence is guaranteed if and only if the spectral radius $\rho(B)$ of B is < 1. The above assumptions may then be considered as sufficient conditions for $\rho(B) < 1$. Usually, $\rho(B)$ will not be known.

6. Iterated Reductions

For certain operators, it is appropriate to perform several reductions. Let us assume that we are given linear operators L_k $(k=1,2,\ldots,n)$ mapping a linear space X_{k-1} (with $X_0 = S$) into a linear space X_k such that all these operators have the properties (2.1), (2.2) (instead of L). Let, moreover,

$$L_k M_{k-1} = M_k - N_k \quad (k=1,2,\ldots,n) \tag{6.1}$$

with $M_0 = M$ and linear operators M_k, N_k mapping R into X_k $(k=1,2,\ldots,n)$. Then, we obtain the following result.

6.2 Theorem. *Suppose that the following three conditions are satisfied:*

(i) *All operators* N_k *are positive* ($u \geq o$ *implies* $N_k u \geq o$).

(ii) $M_n v \not\leq o$ *for all* $v \in R$ *with* $v \geq o$, $v \neq o$.

(iii) *There exists an element* $z \in R$ *such that*

$$z \geqq o, \quad Mz > o, \quad \text{or} \quad z \geqq o, \quad \tilde{M}z > o \quad \text{with} \quad \tilde{M} = L_n \dots L_2 L_1 M.$$

Then, M *has the Properties* **IP** *and* **IP'**.

<u>Proof</u>. We have $LM = M_n - N$ with

$$L = L_n L_{n-1} \dots L_2 L_1 \tag{6.3}$$

$$N = N_n + L_n N_{n-1} + L_n L_{n-1} N_{n-2} + \dots \, . \tag{6.4}$$

The operator $L: S \to X_n$ satisfies (2.1), (2.2), and $N: R \to X_n$ is positive. Thus, the n reductions described by (6.1) can be combined to one single reduction, and this theorem follows from Theorem 3.1.

6.5 <u>Corollary</u>. *Under the assumptions of Theorem* 6.2 *it is also true that, for all* $u \in R$,

$$Mu \geqq o \ \text{implies} \ u \geqq o, \ M_k u \geqq o \ (k=1,2,\dots,n), \tag{6.7}$$
$$Mu > o \ \text{implies} \ u \succ o, \ M_k u > o \ (k=1,2,\dots,n).$$

This is an easy consequence of Theorem 6.2 and the positivity of the operators L_k and N_k.

The relation (6.7) means that M is inverse-positive with respect to the order relation $\leqq$ in S and the order relation in R, given by the set of inequalities $u \geqq o$, $M_k u \geqq o$ $(k=1,2,\dots,n)$. One could very well start with this different order relation in the beginning. We will, however, not pursue this possibility here.

Although iterated reductions as in (6.1) are easily understood as one single reduction with the product operator L in (6.3), the method of iterated reductions seems to be quite fruitful, practically. We will discuss its meaning and its connection with other methods a little further.

If $(R,\leq)$ is partially ordered and $\tilde{M} = L_n \ldots L_2 L_1 M$ is inverse-positive, we have $M^{-1} = \tilde{M}^{-1} L_n \ldots L_2 L_1$. Thus, the method of iterated reductions may be looked at as partly constructing the inverse M^{-1}. Of course, $\tilde{M}^{-1}$ is generally not known, either. What may be known is M_n^{-1} which operator is connected with M^{-1} as follows:

$$M^{-1} = M_n^{-1} L_n \ldots L_2 L_1 + \tilde{N} \quad \text{with} \quad \tilde{N} = M_n^{-1} N M^{-1}$$

and N as in (6.4). Since $\tilde{N}$ is positive (for positive N_k and inverse-positive M_n, M) we have obtained a minorant of M^{-1} in form of a product of positive operators, $M_n^{-1} L_n \ldots L_2 L_1$.

If the operators L_k have inverses P_k, the recursion formulaes (6.1) can be written as

$$\text{(6.8)} \qquad M_{k-1} = P_k M_k - Q_k \quad (k=1,2,\ldots,n)$$

with $Q_k = P_k N_k$. Then, the whole method may be considered as one of splitting M --additively and multiplicatively--into operators P_k, M_k, Q_k as shown by the above formulaes.

To use splitting in order to prove inverse-positivity of differential operators has been suggested earlier, and this has been worked out in detail for ordinary differential operators of the fourth order (see [4] and the references given there). Some of those results have been carried over to differential operators of the third order and operators of higher order by Trottenberg [5]. The proofs in those papers are not the same as here, but related. This paper provides a more transparent approach to reduce the problem for a given operator M to a simpler problem. The following example shows how iterated reductions of type (6.1) can be used for treating a differential operator of the sixth order.

<u>Example</u>. Let L be as in Example 4.5 (with $q(s) \leq 0$), but consider now the boundary conditions

$$u(0) = 0 \qquad , \quad u(1) = 0$$

$$u'''(0) - \alpha u''(0) + \beta u'(0) = 0, \quad -u'''(1) - au''(1) - bu'(1) = 0,$$

$$u^{IV}(0) - \gamma u'(0) \qquad , \quad u^{IV}(1) + cu'(1) = 0$$

with nonnegative α, γ, a, c.

The statement 4.6 holds also in this case.

In order to prove this, define the spaces R,S, the operator M, and the order relations in R,S, as in Example 4.5. The operator $L_1 = A$ with A as given in that example maps S into

$$X_1 = \{x_1 \in C_2[0,1]: x_1(0) = x_1(1) = 0\}.$$

For $x_1 \in X_1$, define $x_1 \geq 0$ to hold pointwise, and $x_1 > o$ by $x_1 \succ o$, i.e. $x_1(s) > 0$ $(0 < s < 1)$, $x_1'(0) > 0$, $x_1'(1) < 0$. Then, L_1 has the properties (2.1), (2.2) and we calculate by partially integrating, $L_1 M = M_1 - N_1$ with

$$M_1 u = u^{IV}, N_1 u = (1-s)u^{IV}(0) + su^{IV}(1) - A(qu).$$

Because of the given boundary conditions and $q(s) \leq 0$, the operator N_1 is positive.

If we can show that this fourth order differential operator $M_1: R \to X_1$ has Property I, the proof is completed. This does not depend on the way this is shown. We will apply a second reduction here.

The operator $L_2 = B$ with B defined in Example 4.5 maps X_1 into

$$X_2 = \{x_2 \in C_4[0,1]: x_2(0) = x_2'(0) = x_2(1) = x_2'(1) = 0\}.$$

For $x_2 \in X_2$, define again $x_2 \geq o$ to hold pointwise and $x_2 > o$ by $x_2 \succ o$, i.e. $x_2(s) > 0$ $(0 < t < 1)$, $x_2''(0) > 0$, $x_2''(1) > 0$. The operator

L_2 satisfies (2.1), (2.2) also, and we obtain $L_2M_1 = M_2 - N_2$ with N_2 the null operator, and

$$(M_2u)(s) = -2s(1-s)u''(s) + 2(1-2s)u'(s) - 2(1-s)^2u'(0) + 2s^2u'(1).$$

It remains to be shown that M_2 has Property I. For that, we could try a third reduction. But, we will now use a direct proof.

If $v \in R$ with $v \geqq o$, $v \not\succ o$, then either $v(t) = 0$ for some $t \in (0,1)$, or $v''(0) = 0$, or $v''(1) = 0$, or several of these relations hold. In the first case, we have $v'(t) = 0$, $v''(t) \geqq 0$ so that $(M_2u)(t) \leqq 0$ and thus, $M_2v \not\succ o$.

In the second case, $v'(0) = 0$, we calculate $(M_2v)'(0) = 0$ and $(M_2v)''(0) = -2v'''(0) + 4v'(1)$. This last expression is nonpositive because of the boundary conditions and $v''(0) \geqq 0$, $v'(1) \leqq 0$. Thus, $M_2v \not\succ o$ is shown again.

The third case, $v'(1) = 1$, is handled correspondingly.

Notice, that these arguments do not carry through if a splitting $L_2M_1 = \tilde{M}_2 - \tilde{N}_2$ with $\tilde{N}_2v = 2(1-s)^2v'(0) - 2s^2v'(1)$ is used.

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